On Some Approaches to the Spectral Excess Theorem for Nonregular Graphs.

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Abstract

The Spectral Excess Theorem (*SPET*) for distance-regular graphs states that a regular (connected) graph is distance-regular if and only if its spectral-excess equals its average excess. Recently, some local or global approaches to the *SPET* have been used to obtain new versions of the theorem for nonregular graphs, and also to study the problem of characterizing the graphs which have the corresponding distance-regularity property. In this paper, some of these versions are related and compared, and some of their results are improved. As a result, a sufficient condition for a graph to be distance-polynomial is obtained.

1 Introduction

The spectral excess theorem [15] (*SPET*) states that a regular (connected) graph Γ is distance-regular if and only if its spectral-excess (a number which can be computed from the spectrum of Γ) equals its average excess (the mean of the numbers of vertices at maximum distance from every vertex), see [8, 14] for short proofs. Since the paper [15] appeared, different approaches (local or global) of the *SPET* have been given. The interest of the inequalities so obtained is the characterization of some kind of distance-regularity, and this happens when equalities are attained.

One of such versions concerns with the so-called pseudo-distance-regularity [17], which is a natural generalization, for nonregular graphs, of the standard distance-regularity [1, 2].

As shown in [17], the price to be paid for speaking about distance-regularity in nonregular graphs is locality. More precisely, what is called "pseudo-distance-regularity" around a vertex. Thus, generalizing a result of Godsil and Shawe-Taylor [21], the author recently proved that when the pseudo-distance-regularity is shared by all the vertices, the graph is either distance-regular or distance-biregular, see [13].

In this paper the commonly used approaches to the *SPET*, that is, local and global, are compared and related. In particular, Lee and Weng [23] recently derived an inequality for nonregular graphs which is similar to the one that leads to the *SPET*, and posed the problem of characterizing the graphs for which the equality is attained. As a result of our study, we show that, in some cases, such an equality is a sufficient condition a the graph Γ to be distance-polynomial, which requires that every distance matrix of Γ is a polynomial in its adjacency matrix (see Weichel [24]).

2 Preliminaries

2.1 Graphs and their spectra

Let us first recall some basic notation and results on which our study is based. For more background on spectra of graphs, distance-regular graphs, and their characterizations, see [1, 2, 3, 5, 9, 12, 20].

Throughout this paper, Γ denotes a (finite, simple and connected) graph with vertex set V, order $n = |V|$, and adjacency matrix $\boldsymbol{A}$. The *distance* between two vertices u and v is denoted by $\operatorname{dist}(u, v)$, so that the *eccentricity* of vertex u is $\varepsilon_u = \max_{v\in V} \operatorname{dist}(u, v)$ and the *diameter* of the graph is $D = \max_{u\in V} \varepsilon_u$. The set of vertices at distance i, from a given vertex $u \in V$ is denoted by $\Gamma_i(u)$, for $i = 0, 1, \ldots, D$, and $N_i(u) = \Gamma_0(u)\cup\Gamma_1(u)\cup\cdots\cup\Gamma_i(u)$. The degree of vertex u is $\delta_u = |\Gamma(u)| = |\Gamma_1(u)|$.

The spectrum of Γ is denoted by $\operatorname{sp}\Gamma = \operatorname{sp}\boldsymbol{A} = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where the different eigenvalues of Γ are in decreasing order, $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, and the superscripts stand for their multiplicities $m_i = m(\lambda_i)$. In particular, note that $m_0 = 1$ (since Γ is connected) and $m_0+m_1+\cdots+m_d = n$. Moreover, λ_0 has a positive eigenvector (usually called the *Perron vector*), denoted by $\boldsymbol{\nu}$ when it is normalized in such a way that its minimum component is 1 [17], and $\boldsymbol{\alpha}$ when the normalization condition is $\|\boldsymbol{\alpha}\|^2 = n$ [23]. For instance, if Γ is regular, we have $\boldsymbol{\nu} = \boldsymbol{\alpha} = \boldsymbol{j}$, the all-1 vector.

Let $\boldsymbol{E}_i$, $i = 0, 1, \ldots, d$, be the idempotents of $\boldsymbol{A}$, that is $\boldsymbol{E}_i = \frac{1}{\phi_i}\prod_{j\neq i}(\boldsymbol{A} - \lambda_j\boldsymbol{I})$, where $\phi_i = \prod_{j\neq i}(\lambda_i - \lambda_j)$. The *u-local multiplicity* of λ_i is $m_u(\lambda_i) = \|\boldsymbol{E}_i\boldsymbol{e}_u\|^2 = (\boldsymbol{E}_i)_{uu}$ and $\operatorname{ev}_u\Gamma$ denotes the set of *u-local eigenvalues*, that is, those eigenvalues of Γ with nonzero u-local multiplicity. Since $\boldsymbol{e}_u$ is unitary, we have $\sum_{i=0}^{d} m_u(\lambda_i) = 1$. We refer to the pair $(\operatorname{ev}_u\Gamma, m_u)$, constituted by the set of u-local eigenvalues and the normalized weight function m_u defined by the u-local multiplicities, as the *u-local spectrum* of Γ. Moreover, as $\boldsymbol{E}_0\boldsymbol{e}_u = \frac{1}{\|\boldsymbol{\alpha}\|^2}\langle\boldsymbol{e}_u, \boldsymbol{\alpha}\rangle\boldsymbol{\alpha} = \frac{\alpha_u}{n}\boldsymbol{\alpha}$, we have that $m_u(\lambda_0) = \frac{\alpha_u^2}{n}$ and, hence, $\lambda_0 \in \operatorname{ev}_u\Gamma$. Let $\operatorname{ev}_u^\star\Gamma = \operatorname{ev}_u\Gamma \setminus \{\lambda_0\}$ and $d_u = |\operatorname{ev}_u^\star\Gamma|$. In [16] it was shown that then the eccentricity of

u satisfies $\varepsilon_u \leq d_u$. When equality is attained, we say that u is an *extremal vertex*.

Recall that, for every $i = 0, 1, \ldots, D$, the distance matrix $\boldsymbol{A}_i$ has entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\operatorname{dist}(u, v) = i$, and $(\boldsymbol{A}_i)_{uv} = 0$ otherwise. In particular, $\boldsymbol{A}_0 = \boldsymbol{I}$ and $\boldsymbol{A}_1 = \boldsymbol{A}$. From the positive (column) eigenvector $\boldsymbol{\alpha}$, $\|\boldsymbol{\alpha}\|^2 = n$, we consider the matrices $\boldsymbol{J}^* = \boldsymbol{\alpha}\boldsymbol{\alpha}^\top$, with entries $(\boldsymbol{J}^*)_{uv} = \alpha_u\alpha_v$ for any $u, v \in V$, and $\boldsymbol{A}_i^* = \boldsymbol{A}_i \circ \boldsymbol{J}^*$, which can viewed as a "weighted" version of the distance matrix $\boldsymbol{A}_i$. In fact, the approach of giving weights (which are the entries of the Perron vector) to the vertices of a nonregular graph has been recently often used in the literature (see, for instance, [17, 15, 16, 10, 23]).

2.2 Predistance polynomials

In the vector space of symmetric matrices $\mathbb{R}^{n\times n}$, define the scalar product

$$\langle \boldsymbol{M}, \boldsymbol{N}\rangle = \frac{1}{n}\operatorname{tr}(\boldsymbol{M}\boldsymbol{N}) = \frac{1}{n}\sum_{i,j}(\boldsymbol{M}\circ\boldsymbol{N})_{ij}$$

in such a way that, within the adjacency algebra of a graph Γ with adjacency matrix $\boldsymbol{A}$ and spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \ldots, \lambda_d^{m(\lambda_d)}\}$, we can consider the following scalar product in $\mathbb{R}^n[x]/(Z)$ (where Z is the minimum polynomial of $\boldsymbol{A}$):

$$\langle p, q\rangle_\Gamma = \frac{1}{n}\operatorname{tr}(p(\boldsymbol{A})q(\boldsymbol{A})) = \langle p(\boldsymbol{A}), q(\boldsymbol{A})\rangle = \frac{1}{n}\sum_{i=0}^{d} m(\lambda_i)p(\lambda_i)q(\lambda_i) \qquad p, q \in \mathbb{R}_d[x]. \quad (1)$$

Then, the *predistance polynomials* $p_0, p_1, \ldots, p_d$, introduced in [15] and extensively used in subsequent papers (they were first called the "proper polynomials"; their present name was first proposed in [12]), are the orthogonal polynomials with respect to the product $\langle\cdot,\cdot\rangle_\Gamma$, normalized in such a way that $\|p_i\|_\Gamma^2 = p_i(\lambda_0)$. (This makes sense since, as it is well-known from the theory of orthogonal polynomials, $p_i(\lambda_0) > 0$ for every $i = 0, 1, \ldots, d$, see, for instance, [4, 8].) As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence with coefficients being the *preintersection numbers*, a concept studied in [6].

There is a local version of the predistance polynomials which goes as follows. Given a vertex u with $d_u + 1$ different local eigenvalues, we can consider the u-local scalar product

$$\langle p, q\rangle_u = (p(\boldsymbol{A})q(\boldsymbol{A}))_{uu} = \langle p(\boldsymbol{A})\boldsymbol{e}_u, q(\boldsymbol{A})\boldsymbol{e}_u\rangle = \sum_{i=0}^{d} m_u(\lambda_i)p(\lambda_i)q(\lambda_i), \quad p, q \in \mathbb{R}_{d_u}[x], \quad (2)$$

where we have used that $\boldsymbol{e}_u = \sum_{i=0}^{d}\boldsymbol{E}_i\boldsymbol{e}_u$ and $p(\boldsymbol{A})\boldsymbol{E}_i = p(\lambda_i)\boldsymbol{E}_i$. (Notice that the sum has at most d_u nonzero terms.) Then, the *u-local predistance polynomials* $\{p_i^u\}_{0\leq i\leq d_u}$ are the orthogonal sequence with respect to the product $\langle\cdot,\cdot\rangle_u$, but now they are normalized in such a way that $\|p_i^u\|_u^2 = \alpha_u^2 p_i^u(\lambda_0)$. Notice that, since $\sum_{u\in V} m_u(\lambda_i) = m(\lambda_i)$, the above global scalar product (1) is the mean over V of the local products in (2):

$$\langle p, q\rangle_\Gamma = \frac{1}{n}\sum_{u\in V}\langle p, q\rangle_u. \quad (3)$$

2.3 Hoffman polynomials

The *sum predistance polynomials* q_j, for $0 \leq j \leq d$, are defined as $q_j = p_0 + p_1 + \cdots + p_j$, so that the *Hoffman polynomial* H [22], characterized by $H(\lambda_i) = n\delta_{0i}$ for $0 \leq i \leq d$, turns out to be $H = q_d$ [4], and satisfies $H(\boldsymbol{A}) = \boldsymbol{J}$ if and only if Γ is regular.

Recently, some general (local or global) versions of the Hoffman polynomial has been considered (see, for instance, [17, 15, 23]). Thus, with $\boldsymbol{\nu}$ being any Perron vector, the *preHoffman polynomial* is defined as

$$H = p_0 + p_1 + \cdots + p_d = \frac{\|\boldsymbol{\nu}\|^2}{\pi_0} \prod_{i=0}^{d} (x - \lambda_i),$$

where $\pi_0 = \prod_{i=0}^{d} (\lambda_0 - \lambda_i)$, and satisfies

$$(H(\boldsymbol{A}))_{uv} = \nu_u \nu_v \qquad (u, v \in V).$$

Similarly, if $\mathrm{ev}_u\, \Gamma = \{\mu_0 (= \lambda_0), \mu_1, \ldots, \mu_{d_u}\}$, the *$u$-local preHoffman polynomial* is

$$H^u = p_0^u + p_1^u + \cdots + p_{d_u}^u = \frac{\|\boldsymbol{\nu}\|^2}{\pi_0} \prod_{i=0}^{d_u} (x - \mu_i),$$

where $\pi_0 = \prod_{i=1}^{d_u} (\lambda_0 - \mu_i)$. Note that $H^u(\boldsymbol{A})\boldsymbol{e}_u = H(\boldsymbol{A})\boldsymbol{e}_u$ since $\boldsymbol{e}_u$ has nonzero projections only on $\ker(\boldsymbol{A} - \mu_i \boldsymbol{I})$, $i = 0, 1, \ldots, d_u$.

2.4 Distance-regularity around a vertex

Let $\Gamma = (V, E)$ be a graph with adjacency matrix $\boldsymbol{A}$, maximum eigenvalue λ_0 and Perron vector $\boldsymbol{\alpha}$, $\|\boldsymbol{\alpha}\|^2 = n$. Consider the map $\boldsymbol{\rho} : V \longrightarrow \mathbb{R}^n$ defined by $\boldsymbol{\rho}(u) = \boldsymbol{\rho}_u = \alpha_u \boldsymbol{e}_u$, where $\boldsymbol{e}_u$ is the coordinate vector. Note that, since $\|\boldsymbol{\rho}_u\| = \alpha_u$, we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of Γ. In doing so we "regularize" the graph, in the sense that the *average weighted degree* of each vertex $u \in V$ becomes a constant:

$$\delta_u^* = \frac{1}{\alpha_u} \sum_{v \in \Gamma(u)} \alpha_v = \lambda_0. \tag{4}$$

A graph Γ is said to be *pseudo-distance-regular around a vertex* $u \in V$ with eccentricity $\mathrm{ecc}(u) = \varepsilon_u$ (or *u-local pseudo-distance-regular*) if the numbers, defined for any vertex $v \in \Gamma_i(u)$,

$$c_i^*(v) = \frac{1}{\alpha_v} \sum_{w \in \Gamma_{i-1}(u)} \alpha_w, \quad a_i^*(v) = \frac{1}{\alpha_v} \sum_{w \in \Gamma_i(u)} \alpha_w, \quad b_i^*(v) = \frac{1}{\alpha_v} \sum_{w \in \Gamma_{i+1}(u)} \alpha_w, \tag{5}$$

depend only on the value of $i = 0, 1, \ldots, \varepsilon_u$. In this case, we denote them by c_i^*, a_i^*, and b_i^*, respectively, and they are referred to as the *u-local pseudo-intersection numbers* of Γ. In particular, when Γ is regular, $\boldsymbol{\alpha} = \boldsymbol{j}$ and the u-local pseudo-distance-regularity coincides with the distance-regularity around u (see Brouwer, Cohen and Neumaier [2]).

3 The *SPET* for nonregular graphs

The following result is stated in terms of the *sum u-local predistance polynomials* $q_j^u = p_0^u + p_1^u + \cdots + p_j^u$.

Proposition 3.1 ([16]) *Let u be a vertex of a graph, with u-local predistance polynomials $\{p_i^u\}_{0\le i\le d_u}$ and let $q_j^u = \sum_{i=0}^{j} p_i$. Then, for any polynomial $r \in \mathbb{R}_j[x]$, $0 \le j \le d_u$, we have*

$$\frac{r(\lambda_0)}{\|r\|_u} \le \frac{1}{\alpha_u}\|\boldsymbol{\rho}_{N_j(u)}\|, \tag{6}$$

and equality holds if and only if u is extremal and

$$\frac{1}{\|r\|_u} r(\boldsymbol{A})\boldsymbol{e}_u = \boldsymbol{e}_{N_j(u)}. \tag{7}$$

In this case, $r = \eta q_j^u$ for any $\eta \in \mathbb{R}$, whence (6) *and* (7) *become, respectively,*

$$q_j^u(\lambda_0) = \|\boldsymbol{\rho}_{N_j(u)}\|^2, \tag{8}$$

and

$$q_j^u(\boldsymbol{A})\boldsymbol{e}_u = \alpha_u \boldsymbol{\rho}_{N_j(u)} \quad \iff \quad (q_j^u(\boldsymbol{A}))_{uv} = \begin{cases} \alpha_u\alpha_v & \textit{if } \operatorname{dist}(u,v) \le j, \\ 0 & \textit{otherwise.} \end{cases} \tag{9}$$

As a consequence, the following characterizations of pseudo-distance-regularity around a vertex were also proved by the same authors, where (b) can be seen as a *Local Spectral Excess Theorem.*

Theorem 3.2 ([16]) *Let $\Gamma = (V,E)$ be a connected graph and let u be a vertex with eccentricity ε_u. Then, Γ is pseudo-distance-regular around u if and only if any of the following conditions holds:*

(a) *There exist polynomials $\{r_i\}_{0\le i\le\varepsilon_u}$ with $\operatorname{dgr} r_i = i$ such that*

$$r_i(\boldsymbol{A})\boldsymbol{e}_u = \alpha_u \boldsymbol{\rho}_{\Gamma_i(u)}.$$

If this is the case, u is extremal, $\varepsilon_u = d_u$, and $r_i = p_i^u$ for $i = 0,1,\ldots,d_u$.

(b) *The u-local predistance polynomial with highest degree satisfies*

$$p_{d_u}^u(\lambda_0) = \|\boldsymbol{\rho}_{\Gamma_{d_u}(u)}\|^2.$$

Let $\delta_D^* = \|\boldsymbol{A}_D^*\|^2$ be the arithmetic mean of the numbers $\alpha_u^2 \sum_{v\in\Gamma_D(u)} \alpha_v^2$ for $u \in V$, which is called the *average weighted excess*, and let $p_{\ge D}(\lambda_0) = (q_d - q_{D-1})(\lambda_0) = n - q_{D-1}(\lambda_0)$ be the so-called *generalized spectral excess*. Then, Lee and Weng [23] proved the following version of the spectral excess theorem for nonregular graphs.

Theorem 3.3 ([23]) *Let Γ be a connected graph with n vertices, diameter D, weight distance matrices $\boldsymbol{I}^*, \boldsymbol{A}^*, \ldots, \boldsymbol{A}_D^*$, distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$, predistance polynomials $p_0, p_1, \ldots, p_d$, and positive eigenvalue $\boldsymbol{\alpha}$. Then,*

$$\delta_D^* \leq p_{\geq D}(\lambda_0), \tag{10}$$

and equality is attained if and only if

$$\boldsymbol{A}_D^* = p_{\geq D}(\boldsymbol{A}). \tag{11}$$

The proof of this theorem is basically the same as the corresponding one in [14] (the only difference is the use of "weights" on the vertices). In [23] Lee and Weng posed the problem of characterizing the graphs for which the equality in (10) is attained, and gave an answer for two kinds of regular graphs. Namely, when $D = d$, in which case the graphs are distance-regular; and when $D = 2$, where the graphs turn to be distance-polynomial [24]. In fact, both cases had been already covered by the standard spectral excess theorem and some of its generalizations [7].

Now, we use the local approach of Proposition 3.1 to obtain an improvement of the bound in (11), which leads to new characterizations of some distance-regularity properties. We begin with the following more general result, where we use the sums of the weighted distance matrices $\boldsymbol{S}_j^* = \sum_{i=0}^{j} \boldsymbol{A}_i^*$, $j = 0, 1, \ldots, D$.

Theorem 3.4 *Let Γ be a graph and, for some integer $j \leq \min\{d_u : u \in V\}$, let $H_{\leq j}^*$ be the harmonic mean of the numbers $\frac{1}{\alpha_u^2}\|\boldsymbol{\rho}_{N_j(u)}\|^2$ for $u \in V$. Then,*

$$q_j(\lambda_0) \leq H_{\leq j}^*,$$

and equality holds if and only if

$$q_j(\boldsymbol{A}) = \boldsymbol{S}_j^*.$$

Proof. With $r = q_j$, the inequality (6) of Proposition 3.1 can be written as

$$\|q_j\|_u^2 \geq \frac{\alpha_u^2 q_j(\lambda_0)^2}{\|\boldsymbol{\rho}_{N_j(u)}\|^2}. \tag{12}$$

Thus, by taking the average over all vertices, we have

$$\frac{q_j(\lambda_0)^2}{n} \sum_{u \in V} \frac{\alpha_u^2}{\|\boldsymbol{\rho}_{N_j(u)}\|^2} \leq \frac{1}{n} \sum_{u \in V} \|q_j\|_u^2 = \|q_j\|_\Gamma^2 = q_j(\lambda_0),$$

where we used (3). Consequently,

$$q_j(\lambda_0) \leq \frac{n}{\sum_{u \in V} \frac{\alpha_u^2}{\|\boldsymbol{\rho}_{N_j(u)}\|^2}} = H_{\leq j}^* \tag{13}$$

Besides, equality can only hold if and only if all the inequalities in (12) are also equalities and, hence, $q_j = \eta_u q_j^u$ for every vertex u and respective constant η_u. Consequently, from (9),

$$q_j(\boldsymbol{A})\boldsymbol{e}_u = \eta_u \alpha_u \boldsymbol{\rho}_{N_j(u)} \qquad \Longleftrightarrow \qquad (q_j(\boldsymbol{A}))_{uv} = \begin{cases} \eta_u \alpha_u \alpha_v & \text{if } \operatorname{dist}(u,v) \le j, \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

To show that the constant η_u does not depend on the vertex u, let u, v be two adjacent vertices. Then, by (14),

$$\eta_u \alpha_u \alpha_v = (q_j(\boldsymbol{A}))_{uv} = (q_j(\boldsymbol{A}))_{vu} = \eta_v \alpha_v \alpha_u,$$

and, hence $\eta_u = \eta_v$. Therefore, since Γ is connected, we get the claimed result. □

A graph Γ with diameter D is called *m-partially distance-regular*, for some $0 \le m \le D$, if their predistance polynomials satisfy $p_i(\boldsymbol{A}) = \boldsymbol{A}_i$ for every $i \le m$, see [7]. It is immediate to show that every m-partially distance-regular with $m \ge 2$ must be regular. By using basically the same proof in [7, Prop. 2.5], we have the following result.

Proposition 3.5 *Let Γ be a graph with $d+1$ distinct eigenvalues, sum predistance polynomials q_j, and sums of weighted distance matrices $\boldsymbol{S}_j^*$, $j = 0, 1, \ldots, D$. Then, Γ is m-partially distance-regular if and only if*

$$q_j(\boldsymbol{A}) = \boldsymbol{S}_j^* \qquad \textit{for} \qquad j = m-1, m. \tag{15}$$

In fact, in our "weighted case" the proof gives that $p_i(\boldsymbol{A}) = \boldsymbol{A}_i^*$ for every $i = 0, 1, \ldots, m$ and, hence, we conclude that the graph must be regular since $p_0(\boldsymbol{A}) = \boldsymbol{I} = \boldsymbol{I}^*$.

As an immediate consequence of the above and Theorem 3.4 we have:

Proposition 3.6 *Let Γ be a graph and, as above, let $H_{\le j}^*$ be the harmonic mean of the numbers $\frac{1}{\alpha_u^2}\|\boldsymbol{\rho}_{N_j(u)}\|^2$ for $u \in V$. Then,*

$$(q_{m-1} + q_m)(\lambda_0) \le H_{\le m-1}^* + H_{\le m}^*, \tag{16}$$

and equality holds if and only if Γ is regular and m-partially distance-regular

As another corollary of Theorem 3.4, by taking $j = D-1$ and considering that $p_{\ge D} = H - q_{D-1}$, we get the following result, which is an improvement of Theorem 3.3.

Theorem 3.7 *Let Γ be a graph with $H_{\le D-1}^*$ being the harmonic mean of the numbers $\frac{1}{\alpha_u^2}\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2$ for $u \in V$, and let δ_D^* be the arithmetic mean of the numbers $\alpha_u^2\|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2$. Then,*

$$p_{\ge D}(\lambda_0) \overset{(i)}{\ge} n - H_{\le D-1}^* \overset{(ii)}{\ge} \delta_D^*. \tag{17}$$

Moreover, equality in (i) holds if and only if

$$p_{\ge D}(\boldsymbol{A}) = \boldsymbol{A}_D^*, \tag{18}$$

whereas equality in (ii) occurs when the numbers $\|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2$ are the same for all $u \in V$.

Proof. Inequality (i) and the corresponding case of equality are consequences of Theorem 3.4 with $k = D-1$:

$$p_{\geq D}(\lambda_0) = n - q_{D-1}(\lambda_0) \geq n - H^*_{\leq D-1} = n - \frac{n}{\sum_{u\in V} \frac{\alpha_u^2}{\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2}}. \tag{19}$$

In order to prove (ii), we have that $\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2 = \|\boldsymbol{\alpha}\|^2 - \|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2 = n - \|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2$. Then,

$$\begin{aligned} \delta^*_D &= \frac{1}{n}\sum_{u\in V} \alpha_u^2 \|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2 = \frac{1}{n}\sum_{u\in V}\alpha_u^2(n - \|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2) \\ &= n - \frac{1}{n}\sum_{u\in V}\alpha_u^2\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2. \end{aligned} \tag{20}$$

Thus, from (19) and (20) we see that $n - H^*_{\leq D-1} \geq \delta^*_D$ if and only if

$$n^2 \leq \sum_{u\in V}\alpha_u^2\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2 \sum_{u\in V}\frac{\alpha_u^2}{\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2}$$

or, using that $\|\boldsymbol{\alpha}\|^2 = \sum_{u\in V}\alpha_u^2 = n$,

$$\begin{aligned} \sum_{u\in V}\alpha_u^2\sum_{v\in V}\alpha_v^2 &= \sum_{u\in V}\alpha_u^4 + \sum_{\{u,v\}\subset V} 2\alpha_u^2\alpha_v^2 \\ &\leq \sum_{u\in V}\alpha_u^2\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2 \sum_{v\in V}\frac{\alpha_v^2}{\|\boldsymbol{\rho}_{N_{D-1}(v)}\|^2} \\ &= \sum_{u\in V}\alpha_u^4 + \sum_{\{u,v\}\subset V}\alpha_u^2\alpha_v^2\left(\frac{\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2}{\|\boldsymbol{\rho}_{N_{D-1}(v)}\|^2} + \frac{\|\boldsymbol{\rho}_{N_{D-1}(v)}\|^2}{\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2}\right), \end{aligned}$$

which is always true because, for any positive real numbers r, s, we have $\frac{r}{s} + \frac{s}{r} \geq 2$ with equality if and only if $r = s$. Therefore, equality in (ii) happens if and only if $\|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2 = \|\boldsymbol{\rho}_{N_{D-1}(v)}\|^2$ for all $u, v \in V$, and the result follows from the fact that $\|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2 = n - \|\boldsymbol{\rho}_{N_{D-1}(u)}\|^2$ for every $u \in V$. □

Notice that the invariance of the numbers $\|\boldsymbol{\rho}_{\Gamma_D(u)}\|^2$ follows also from (18) by multiplying both terms by $\boldsymbol{\alpha}$, and the common value is $p_{\geq D}(\lambda_0)$.

To compare the above bounds for $p_{\geq D}(\lambda_0)$, let us consider the complete bipartite graph $K_{2,3}$, with spectrum $\operatorname{sp} K_{2,3} = \{\sqrt{6}, 0, -\sqrt{6}\}$ and Perron vector

$$\boldsymbol{\alpha} = (\sqrt{5}/2, \sqrt{5}/2, \sqrt{5}/\sqrt{6}, \sqrt{5}/\sqrt{6}, \sqrt{5}/\sqrt{6})^\top,$$

whence we obtain

$$p_{\geq D}(\lambda_0) = \frac{3}{2} > n - H^*_{D-1} = \frac{25}{17} > \delta^*_D = \frac{35}{24}.$$

As a consequence of Theorem 3.7, we can now given a partial answer to the above mentioned problem posed in [23].

Theorem 3.8 *Let Γ be a graph with arithmetic means δ_i^* of the numbers $\alpha_u^2\|\boldsymbol{\rho}_{\Gamma_i(u)}\|^2$, $i = D-1, D$, and polynomials $p_{\geq D}$ and p_{D-1} satisfying.*

$$\delta_D^* = p_{\geq D}(\lambda_0) \quad \text{and} \quad \delta_{D-1}^* = p_{D-1}(\lambda_0).$$

Then Γ is distance-polynomial.

Proof. By Theorem 3.3, we have that $p_{\geq D}(\boldsymbol{A}) = \boldsymbol{A}_D^*$. Moreover, from the hypotheses

$$\begin{aligned} q_{D-1}(\boldsymbol{A}) &= (H - p_{\geq D})(\boldsymbol{A}) = \boldsymbol{J}^* - \boldsymbol{A}_D^* = \boldsymbol{S}_{D-1}^*, \\ q_{D-2}(\boldsymbol{A}) &= (H - p_{\geq D} - p_{D-1})(\boldsymbol{A}) = \boldsymbol{S}_{D-1}^* - \boldsymbol{A}_{D-1}^* = \boldsymbol{S}_{D-2}^*. \end{aligned}$$

Hence, by Proposition 3.5, Γ is regular and $(D-1)$-partially distance-regular. Consequently, $p_{\geq D}(\boldsymbol{A}) = \boldsymbol{A}_D$ and Γ is distance-polynomial, as claimed. $\Box$

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